Structured power functions

2009 J. Phys. A: Math. Theor. 42 135205

(http://iopscience.iop.org/1751-8121/42/13/135205)

J. Phys. A: Math. Theor. **42** (2009) 135205 (13pp) doi:10.1088/1751-8113/42/13/135205

Structured power functions

Yongbin Chang

Department of Chemistry, University of British Columbia, Vancouver, BC V6T 1Z1, Canada

E-mail: yongbin@chem.ubc.ca

Received 25 October 2008, in final form 5 February 2009
Published 6 March 2009
Online at stacks.iop.org/JPhysA/42/135205

Abstract
In the kinetic theory of gases and plasmas, many integrals associated with Maxwellian velocity distribution such as collision frequencies, Rosenbluth's super potentials and arbitrary order of Fokker–Planck coefficients, can be reduced to the integrals

$$R_n^{(\ell)}(u) = \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^1 \left[\mathrm{e}^{-(y-ux)^2} \pm \mathrm{e}^{-(y+ux)^2}\right] y^n x^\ell \,\mathrm{d}x\,\mathrm{d}y.$$

The usual approach to calculating the integrals is very tedious, particularly for large n and ℓ, and the results expressed in common elemental functions, including power, exponential and error functions, are complex. In this paper, $R_n^{(\ell)}(u)$ are calculated for arbitrary n and ℓ. The properties of the functions $R_n^{(\ell)}(u)$ are studied in detail and systematically. It was found that these functions have very simple properties. In many ways, they are much like common power functions u^n. Thus, $R_n^{(\ell)}(u)$ are referred to as structured power functions in this paper. With structured power functions, many physical results of *arbitrary* order can be presented in simple and meaningful expressions.

PACS numbers: 51.10.+y, 52.25.Dg, 05.20.Dd
Mathematics Subject Classification: 33C99

1. Introduction

Collision frequency in kinetic gas theory [1], and friction and diffusion coefficients in plasma physics are well-known physical results [2]. A recently developed unified kinetic theory [3, 4] postulates that these physical results as well as the arbitrary order of Fokker–Planck coefficients for both Coulomb and hard sphere interactions can be calculated and expressed in a uniform expression. The uniform formula is expressed in the form of a set of functions $R_n^{(\ell)}(u)$. The difference between Coulomb interactions and hard sphere interactions for collision frequencies and the other Fokker–Planck coefficients is characterized by the 'power' n in $R_n^{(\ell)}(u)$. Therefore, the functions $R_n^{(\ell)}(u)$ represent the common properties shared by the multiple physical results from different interactions. However, the relation of these physical results and their common properties has long been obscured by their complicated expressions in the form of elemental functions including exponential, error, positive power and negative power functions. When collision frequencies and the other Fokker–Planck coefficients are expressed in elemental functions, one can hardly see any relations between these physical results for Coulomb and hard sphere interactions. In terms of the structured power functions $R_n^{(\ell)}(u)$, these physical results for hard sphere interactions are always four orders higher in 'power' than those for Coulomb interactions.

In the kinetic theory for gases and plasmas, it is essential to study a test particle's movement when it is in a Maxwellian background, a gas or plasma that is in thermal equilibrium. Many physical phenomena are defined in terms of integrations where the Maxwellian velocity distribution function is employed. For example, the collision frequency is [1]

$$\nu(v) = \int f_M(v_b) g\sigma(g,\theta)\,d\Omega\,d\mathbf{v}_b, \tag{1}$$

where $f_M(v_b)$ is the Maxwellian velocity distribution function for the background particles, $g = |\mathbf{v} - \mathbf{v}_b|$ is the relative speed of a test particle of velocity $\mathbf{v}$ and a background particle $\mathbf{v}_b$, $\sigma(g,\theta)$ is the differential scattering cross section, $d\Omega = \sin\theta\,d\theta\,d\phi$ is the differential of the solid scattering angle.

The relative speed moments of Maxwellian distribution are defined by [5]

$$M_n(v) = \int f_M(v_b)|\mathbf{v} - \mathbf{v}_b|^n\,d\mathbf{v}_b. \tag{2}$$

In plasma physics [6, 7] the relative speed moments of lower orders are called Rosenbluth's super potentials. The relative speed moments are the extension of Rosenbluth's super potentials to arbitrary high orders.

A test particle changes its velocity $\mathbf{v}$ due to random collisions with background particles. The velocity change $\Delta\mathbf{v} = \mathbf{v}' - \mathbf{v}$ describes a single collision event. Here, $\mathbf{v}$ and $\mathbf{v}'$ are the velocities of the test particle before and after a collision. In plasma physics the average effects are called Fokker–Planck coefficients [2, 6, 8],

$$\langle \Delta\mathbf{v}^n \rangle = \int \Delta\mathbf{v}^n f_M(v_b) g\sigma\,d\Omega\,d\mathbf{v}_b, \tag{3}$$

where $\Delta\mathbf{v}^n$ is the nth-order tensor of velocity change.

The energy transfer of a test particle in a single collision is $\frac{m}{2}(v'^2 - v^2)$. The transfer moments are defined by the integral [9, 10]

$$G_n(v) = \int \left[\frac{m}{2}(v'^2 - v^2)\right]^n f_M(v_b) g\sigma\,d\Omega\,d\mathbf{v}_b. \tag{4}$$

It can be proved that all the four integrals (equations (1)–(4)) can eventually be reduced to the integrals [4]

$$R_n^{(\ell)}(u) = \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^1 \left[e^{-(y-ux)^2} \pm e^{-(y+ux)^2}\right] y^n x^\ell\,dx\,dy, \tag{5}$$

where $u = v/v_{th}$ is the dimensionless speed of the test particle, and v_{th} is the thermal velocity of the background particles. The purpose of this paper is to calculate equation (5) and to analyze the properties of the functions $R_n^{(\ell)}(u)$.

It is notable that equation (5) defines two different integrals since the sign between the two exponential functions can be either positive + or negative −. It is easy to ascertain that $R_n^{(\ell)}(-u) = \pm R_n^{(\ell)}(u)$. For any n and ℓ, there is an even and odd function for $R_n^{(\ell)}(u)$. The parity of the functions plays an important role in solving and understanding them. For solving the integrals, the functions $R_n^{(\ell)}(u)$ are grouped into $q_n^{(\ell)}(u)$ and $\overline{q}_n^{(\ell)}(u)$ in section 2. The over line indicates that $\overline{q}_n^{(\ell)}(u)$ have an opposite parity from $q_n^{(\ell)}(u)$. The complete sets of $R_n^{(\ell)}(u)$, $q_n^{(\ell)}(u)$ and $\overline{q}_n^{(\ell)}(u)$ are obtained and expressed in the form of confluent hypergeometric functions. In section 3, we show that the functions $R_n^{(\ell)}(u)$ can also be divided into subsets of polynomials $p_n^{(\ell)}(u)$ and non-polynomials $\overline{p}_n^{(\ell)}(u)$. Of course, a non-polynomial $\overline{p}_n^{(\ell)}(u)$ also has an opposite parity from the polynomial $p_n^{(\ell)}(u)$. At infinity $u \to \infty$, we prove that $R_n^{(\ell)}(u)$ always approach polynomials $p_n^{(\ell)}(u)$. In order to analyze the properties of the functions near $u = 0$, the functions $R_n^{(\ell)}(u)$ are grouped into odd functions $o_n^{(\ell)}(u)$ and even functions $\overline{o}_n^{(\ell)}(u)$ in section 4. The expansions of $\overline{p}_n^{(\ell)}(u)$ at $u = 0$ are obtained. Examples and physical applications are provided in section 5. The arbitrary high-order Fokker–Planck coefficients, the speed moments and the transfer moments are expressed in structured power functions. A discussion and summary are presented in section 6.

2. The functions $q_n^{(\ell)}(u)$ and $\overline{q}_n^{(\ell)}(u)$

In the kinetic theories of gases and plasmas, the integrals (equation (5)) and their equivalents are frequently encountered. Direct evaluation of the integrals for large n and ℓ is extremely tedious but presents no difficulties in principle. The calculation of the equivalent integrals of equation (5) for the *arbitrary* integers n and ℓ appears in the calculation of arbitrary high-order Fokker–Planck coefficients for plasmas [3, 8]. Although improvements were made by Chang [4], the evaluation of the integrals $R_n^{(\ell)}(u)$ is still tedious and cumbersome. In this section, a new approach is presented, one that produces a simple result. The functions $R_n^{(\ell)}(u)$ can be regrouped into two functions of opposite parities:

$$q_n^{(\ell)}(u) = \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^1 \left[\mathrm{e}^{-(y-ux)^2} + (-1)^\ell \mathrm{e}^{-(y+ux)^2}\right] y^n x^\ell \, \mathrm{d}x \, \mathrm{d}y \tag{6}$$

and

$$\overline{q}_n^{(\ell)}(u) = \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^1 \left[\mathrm{e}^{-(y-ux)^2} - (-1)^\ell \, \mathrm{e}^{-(y+ux)^2}\right] y^n x^\ell \, \mathrm{d}x \, \mathrm{d}y. \tag{7}$$

It is easy to establish that $q_n^{(\ell)}(-u) = (-1)^\ell q_n^{(\ell)}(u)$. The parity of $q_n^{(\ell)}(u)$ is determined by the associated index ℓ. When ℓ is even/odd, $q_n^{(\ell)}(u)$ is an even/odd function. The function $\overline{q}_n^{(\ell)}(u)$ has an opposite parity to that of $q_n^{(\ell)}(u)$. The two functions $q_n^{(\ell)}(u)$ and $\overline{q}_n^{(\ell)}(u)$ form a complete set of the functions $R_n^{(\ell)}(u)$.

We first calculate the derivative of $q_n^{(\ell)}(u)$. From

$$\frac{\mathrm{d}}{\mathrm{d}u}\left[\mathrm{e}^{-(y-ux)^2} + (-1)^\ell \mathrm{e}^{-(y+ux)^2}\right] = -x \frac{\mathrm{d}}{\mathrm{d}y}\left[\mathrm{e}^{-(y-ux)^2} + (-1)^{\ell+1} \, \mathrm{e}^{-(y+ux)^2}\right], \tag{8}$$

we obtain the derivative of equation (6)

$$\frac{\mathrm{d}}{\mathrm{d}u} q_n^{(\ell)}(u) = -\frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^1 x \frac{\mathrm{d}}{\mathrm{d}y}\left[\mathrm{e}^{-(y-ux)^2} + (-1)^{\ell+1} \, \mathrm{e}^{-(y+ux)^2}\right] y^n x^\ell \, \mathrm{d}x \, \mathrm{d}y. \tag{9}$$

Taking the partial integration over y in equation (9), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}u} q_n^{(\ell)}(u) = n \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^1 \left[\mathrm{e}^{-(y-ux)^2} + (-1)^{\ell+1} \, \mathrm{e}^{-(y+ux)^2}\right] y^{n-1} x^{l+1} \, \mathrm{d}x \, \mathrm{d}y. \tag{10}$$

Comparing equation (10) with the definition of $q_n^{(\ell)}(u)$ in equation (6), we have

$$\frac{\mathrm{d}}{\mathrm{d}u} q_n^{(\ell)}(u) = n q_{n-1}^{(\ell+1)}(u). \tag{11}$$

This derivative relation (equation (11)) is one of the most important properties of the functions $q_n^{(\ell)}(u)$. If we consider n as the power of the function $q_n^{(\ell)}(u)$, the derivative formula (equation (11)) is similar to the derivative of the regular power functions $\frac{\mathrm{d}}{\mathrm{d}u}u^n = nu^{n-1}$. Using the same method, one can prove that the derivative of $\overline{q}_n^{(\ell)}(u)$ follows the same rule. With the derivative property, the calculation of $q_n^{(\ell)}(u)$ can be reduced to the calculation of $q_n^{(0)}(u)$ since

$$q_n^{(\ell)}(u) = \frac{n!}{(n+\ell)!}\frac{\mathrm{d}^\ell}{\mathrm{d}u^\ell}q_{n+\ell}^{(0)}(u). \tag{12}$$

It is clear that $q_n^{(0)}(u)$ is always an even function. Making a direct integration over x in equation (6) for $\ell = 0$, we obtain

$$q_n^{(0)}(u) = \int_0^\infty \frac{\mathrm{Erf}(u-y) + \mathrm{Erf}(u+y)}{u} y^n \, \mathrm{d}y, \tag{13}$$

where Erf is the error function. Making a partial integration in equation (13) followed by the symbolic integration with the software Mathematica [11], we obtain

$$q_n^{(0)}(u) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}\, {}_1F_1\left(-\frac{n}{2}, \frac{3}{2}; -u^2\right), \tag{14}$$

where Γ is the gamma function and ${}_1F_1$ is the confluent hypergeometric function [11]. Combining equation (14) with (12), we obtain

$$q_n^{(\ell)}(u) = \frac{\Gamma\left(\frac{n+\ell+1}{2}\right)n!}{\Gamma\left(\frac{3}{2}\right)(n+\ell)!}\frac{\mathrm{d}^\ell}{\mathrm{d}u^\ell}\, {}_1F_1\left(-\frac{n+\ell}{2}, \frac{3}{2}; -u^2\right). \tag{15}$$

An equivalent result for $q_n^{(\ell)}(u)$ was obtained in [4] using a different approach. However, the expression of $q_n^{(\ell)}(u)$ in [4] is much more complex than that in equation (15).

The calculation of $\overline{q}_n^{(\ell)}(u)$ follows the same procedure as that for $q_n^{(\ell)}(u)$. It was found that

$$\overline{q}_n^{(\ell)}(u) = \frac{\Gamma\left(\frac{n+\ell+2}{2}\right)n!}{\Gamma\left(\frac{3}{2}\right)(n+\ell+1)!}\frac{\mathrm{d}^\ell}{\mathrm{d}u^\ell}\frac{{}_1F_1\left(-\frac{n+\ell+1}{2}, \frac{1}{2}; -u^2\right) - 1}{u}. \tag{16}$$

The functions $\overline{q}_n^{(\ell)}(u)$ have never been obtained previously. Equations (15) and (16) form a complete set of solutions for the functions $R_n^{(\ell)}(u)$. The latter can be divided into different ways to facilitate the study of their properties.

3. The functions $p_n^{(\ell)}(u)$ and $\overline{p}_n^{(\ell)}(u)$

Structured power functions $R_n^{(\ell)}(u)$ are grouped into $q_n^{(\ell)}(u)$ and $\overline{q}_n^{(\ell)}(u)$ functions whose parities are determined by ℓ. The parities of regular power functions u^n are determined by n. Whether the power function u^n is even or odd depends on whether n is an even or odd integer. It would be helpful to regroup $R_n^{(\ell)}(u)$ in a way that their parities are determined by n. We define

$$p_n^{(\ell)}(u) = \frac{2}{\sqrt{\pi}}\int_0^\infty\int_0^1 \left[\mathrm{e}^{-(y-ux)^2} + (-1)^n \,\mathrm{e}^{-(y+ux)^2}\right] y^n x^\ell \, \mathrm{d}x\, \mathrm{d}y \tag{17}$$

and

$$\overline{p}_n^{(\ell)}(u) = \frac{2}{\sqrt{\pi}}\int_0^\infty\int_0^1 \left[\mathrm{e}^{-(y-ux)^2} - (-1)^n \,\mathrm{e}^{-(y+ux)^2}\right] y^n x^\ell \, \mathrm{d}x\, \mathrm{d}y. \tag{18}$$

Based on the definitions of $p_n^{(\ell)}(u)$, $q_n^{(\ell)}(u)$ and $\overline{q}_n^{(\ell)}(u)$, we obtain

$$p_n^{(\ell)}(u) = \frac{1+(-1)^{n+\ell}}{2}q_n^{(\ell)}(u) + \frac{1-(-1)^{n+\ell}}{2}\overline{q}_n^{(\ell)}(u). \tag{19}$$

With regard to equation (19), we found that the function $p_n^{(\ell)}(u)$ is $q_n^{(\ell)}(u)$ when $n+\ell$ is an even number and is $\overline{q}_n^{(\ell)}(u)$ when $n+\ell$ is an odd number. The solution of $\overline{p}_n^{(\ell)}(u)$ can also be constructed the same way from $q_n^{(\ell)}(u)$, and $\overline{q}_n^{(\ell)}(u)$ whenever necessary.

Assuming $n+\ell$ is an even number, the confluent hypergeometric function in the expression for $q_n^{(\ell)}(u)$ in equation (15) can be reduced to Laguerre polynomials based on the relation [12]

$$ {}_1F_1(-k, j+1; x) = \frac{\Gamma(k+1)\Gamma(j+1)}{\Gamma(k+j+1)} L_k^{(j)}(x). \tag{20}$$

In the case of an even $n+\ell$, we obtain

$$ p_n^{(\ell)}(u) = q_n^{(\ell)}(u) = \frac{(\frac{n+\ell}{2})!n!}{(n+\ell+1)!} \frac{2\mathrm{d}^\ell}{\mathrm{d}u^\ell} L_{\frac{n+\ell}{2}}^{(1/2)}(-u^2), \tag{21}$$

which are polynomial functions. When $n+l$ is an odd number, $p_n^{(\ell)}(u)$ is equal to $\overline{q}_n^{(l)}(u)$, which is also a polynomial function. Therefore, $p_n^{(\ell)}(u)$ are always polynomial functions. These functions have a simple form

$$ p_n^{(\ell)}(u) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{2^{1-2i} n!}{i!(n-2i)!} \frac{u^{n-2i}}{n+\ell+1-2i}, \tag{22}$$

where $\lfloor \frac{n}{2} \rfloor$ is the floor function which represents the largest integer equal to or smaller than $n/2$.

It is clear that all the coefficients in equation (22) are positive numbers. The polynomials $p_n^{(\ell)}(u)$ are so simple that the following properties are obvious from equation (22):

(1) The parities of the functions are consistent with their power,

$$ p_n^{(\ell)}(-u) = (-1)^n p_n^{(\ell)}(u). $$

(2) The derivatives of the functions are

$$ \frac{\mathrm{d}}{\mathrm{d}u} p_n^{(\ell)}(u) = n p_{n-1}^{(\ell+1)}(u). $$

(3) The functions are monotonic increasing functions for $u > 0$.

(4) As the associate index ℓ appears only in the denominator of the coefficients in equation (22), it follows that

$$ p_n^{(\ell)}(u) > p_n^{(\ell+1)}(u) \qquad \text{for} \quad u > 0. $$

It is easy to remember the first three properties since the simple power functions u^n have the same properties.

In mathematics, an Appell sequence [13, 14] is any polynomial sequence $\{p_n(x)\}_{n=0,1,2,\ldots}$ satisfying the identity $\frac{\mathrm{d}}{\mathrm{d}x} p_n(x) = n p_{n-1}(x)$ and in which $p_0(x)$ is a non-zero constant. For a particular ℓ, we can form the Appell sequence $\{p_0^{(n+\ell)}(u), p_1^{(n+\ell-1)}(u), \ldots, p_n^{(\ell)}(u)\}$.

For polynomial functions $p_n^{(\ell)}(u)$, the main index n determines the maximum power and the parities of the functions $p_n^{(\ell)}(u)$. The associated index ℓ only changes the coefficients of the polynomial functions. The larger the value ℓ, the smaller the value of the functions for $u > 0$. The leading term is $\frac{2}{n+\ell+1} u^n$. The regular power functions u^n lack the associate index ℓ. This constitutes the main difference between $p_n^{(\ell)}(u)$ and the regular power functions u^n.

Unlike polynomials $p_n^{(\ell)}(u)$, $\overline{p}_n^{(\ell)}(u)$ are not polynomials but more complicated functions. We can obtain the properties of $\overline{p}_n^{(\ell)}(u)$ by comparing them with $p_n^{(\ell)}(u)$. When u is large and approaches positive infinity, the term $\mathrm{e}^{-(y+ux)^2}$ in both equations (17) and (18) is much smaller than $e^{-(y-ux)^2}$. Thus, the term $e^{-(y+ux)^2}$ can be ignored when u approaches positive infinity. In this case, we have

$$\lim_{u\to+\infty} \overline{p}_n^{(\ell)}(u) = p_n^{(l)}(u). \tag{23}$$

Therefore, $\overline{p}_n^{(\ell)}(u)$ and $p_n^{(\ell)}(u)$ converge when u approaches infinity. Thus $\overline{p}_n^{(\ell)}(u)$ behave the same way as the polynomials $p_n^{(\ell)}(u)$ for a large u. However, $\overline{p}_n^{(\ell)}(u)$ have opposite parities vis-a-vis $p_n^{(\ell)}(u)$. Therefore, the parities of the functions $\overline{p}_n^{(\ell)}(u)$ are inconsistent with their power n. This inconsistency makes it impossible to present $\overline{p}_n^{(\ell)}(u)$ in the form of a polynomial. $\overline{p}_n^{(\ell)}(u)$ must be more complex when they are expressed in elemental functions.

The functions $\overline{p}_n^{(\ell)}(u)$ were expressed in confluent hypergeometric functions from equation (15) or equation (16). One can obtain the expression $\overline{p}_n^{(\ell)}(u)$ in elemental functions by replacing the confluent hypergeometric functions with elemental functions. However, this approach is highly complex. Here, we present a better approach based on $p_n^{(\ell)}(u)$. We know that $\overline{p}_n^{(\ell)}(u)$ are simply structured power functions whose parities are opposite to those of $p_n^{(\ell)}(u)$. Therefore, we may construct $\overline{p}_n^{(\ell)}(u)$ from $p_n^{(\ell)}(u)$ and thus show the parity transition more explicitly. The following results are based on calculations of $\overline{p}_n^{(\ell)}(u)$ for different n and ℓ. We found $\overline{p}_n^{(\ell)}(u)$ to have the structure

$$\overline{p}_n^{(\ell)}(u) = p_n^{(\ell)}(u)\mathrm{Erf}(u) + \frac{2e^{-u^2}}{\sqrt{\pi}} r_n^{(\ell)}(u) + \delta_n^{(\ell)}(u), \tag{24}$$

where

$$\delta_n^{(\ell)}(u) = \frac{n!\ell![((-1)^{n+\ell}-1)\mathrm{Erf}(u) + 1 + (-1)^{n+\ell}]}{2^{n+\ell+1}\Gamma\left(\frac{n+\ell+3}{2}\right)(-u)^{\ell+1}}, \tag{25}$$

$$p_n^{(\ell)}(u) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} a(n,\ell,i)u^{n-2i}, \tag{26}$$

$$r_n^{(\ell)}(u) = \sum_{i=0}^{\lfloor \frac{n+\ell}{2} \rfloor} b(n,\ell,i)u^{n-1-2i}, \tag{27}$$

in which

$$a(n,\ell,i) = \frac{2^{1-2i}n!}{i!(n-2i)!}\frac{1}{n+\ell+1-2i}, \tag{28}$$

and $b(n,\ell,i)$ can be determined by the derivative relation $\frac{d}{du}\overline{p}_n^{(\ell)}(u) = n\overline{p}_{n-1}^{(\ell+1)}(u)$. Unfortunately, we failed to derive an explicit expression for $b(n,\ell,i)$. However, $b(n,\ell,i)$ can be obtained from the recurrent relation

$$\begin{aligned} b(n,\ell,0) &= \frac{1}{2}a(n,\ell,0), \\ b(n,\ell,i) &= \frac{1}{2}a(n,\ell,i) + \frac{n+1-2i}{2}b(n,\ell,i-1) - \frac{n}{2}b(n-1,\ell+1,i-1). \end{aligned} \tag{29}$$

In kinetic theory, many physical phenomena are presented in the form of equation (24). The relations between these phenomena are obscured when they are expressed in the form of elemental functions.

There are three components to the functions $\overline{p}_n^{(\ell)}(u)$ when the latter are expressed in the common elemental functions in equation (24): $p_n^{(\ell)}(u)\mathrm{Erf}(u)$, $\frac{2e^{-u^2}}{\sqrt{\pi}}r_n^{(\ell)}(u)$ and $\delta_n^{(\ell)}(u)$.

When u approaches infinity, we find both $\frac{2e^{-u^2}}{\sqrt{\pi}} r_n^{(\ell)}(u)$ and $\delta_n^{(\ell)}(u)$ approach zero. Since $\lim_{u\to+\infty} \mathrm{Erf}(u) = 1$, we can prove equation (23), i.e. $\lim_{u\to+\infty} \overline{p}_n^{(\ell)}(u) = p_n^{(\ell)}(u)$.

We found that the right-hand side of equation (24) does have an opposite parity to $p_n^{(\ell)}(u)$. Since the error function $\mathrm{Erf}(u)$ is odd, i.e. $\mathrm{Erf}(-u) = -\mathrm{Erf}(u)$, the first term has the opposite parity to $p_n^{(\ell)}(u)$. In the second term, e^{-u^2} has an even parity and $r_n^{(\ell)}(u)$ a parity of $(-1)^{n-1}$ that is opposite to $p_n^{(\ell)}(u)$. The parity of the last term, $\delta_n^{(\ell)}(u)$, should be considered under two different conditions. When $n + \ell$ is even, i.e. $(-1)^\ell = (-1)^n$, we obtain

$$\delta_n^{(\ell)}(-u) = \frac{n!\ell!}{2^{n+\ell}\Gamma\left(\frac{n+\ell+3}{2}\right)u^{\ell+1}} = (-1)^{n+1}\delta_n^{(\ell)}(u). \tag{30}$$

When $n + \ell$ is odd, i.e. $(-1)^\ell = (-1)^{n+1}$, we obtain

$$\delta_n^{(\ell)}(-u) = \frac{n!\ell!\,\mathrm{Erf}(u)}{2^{n+\ell}\Gamma\left(\frac{n+\ell+3}{2}\right)u^{\ell+1}} = (-1)^{n+1}\delta_n^{(\ell)}(u). \tag{31}$$

Under any conditions, $\delta_n^{(\ell)}(-u) = (-1)^{n+1}\delta_n^{(\ell)}(u)$, i.e. the third component $\delta_n^{(\ell)}(u)$ has the opposite parity to $p_n^{(\ell)}(u)$. In conclusion, equation (24) has transformed the parity of $p_n^{(\ell)}(u)$ so that we now have

$$\overline{p}_n^{(\ell)}(-u) = (-1)^{n+1}\overline{p}_n^{(\ell)}(u). \tag{32}$$

Because the functions $\overline{p}_n^{(\ell)}(u)$ always have an opposite parity to their power n, the functions $\overline{p}_n^{(\ell)}(u)$ have a complex form when expressed in common elemental functions.

In this section, the functions $\overline{p}_n^{(\ell)}(u)$ have been expressed in the common elemental functions (equation (24)) from which we obtain $\lim_{u\to+\infty} \overline{p}_n^{(\ell)}(u) = p_n^{(\ell)}(u)$. To have a better understanding of $\overline{p}_n^{(\ell)}(u)$, we need to obtain $\lim_{u\to 0} \overline{p}_n^{(\ell)}(u)$. However, equation (24) does not help in finding the properties of $\overline{p}_n^{(\ell)}(u)$ for $u \to 0$. This is another obstacle to presenting physical phenomena in elemental functions. When u approaches zero, we find that equation (24) has many singular terms, e.g. $u^{-(\ell+1)}, \ldots, u^{-\frac{7-(-1)^n}{2}}, u^{-\frac{3-(-1)^n}{2}}$, especially for a large ℓ. In reality, however, the functions $\overline{p}_n^{(\ell)}(u)$ do not have any singularity at $u = 0$. The analysis in the following section indicates that the functions $\overline{p}_n^{(\ell)}(u)$ are well behaved, just like regular power functions near $u = 0$. These singular terms, $u^{-(\ell+1)}, \ldots, u^{-\frac{7-(-1)^n}{2}}, u^{-\frac{3-(-1)^n}{2}}$, cancel each other out so that $\overline{p}_n^{(\ell)}(u)$ have a finite value at $u = 0$. In equation (24), the power functions (singular and non-singular terms) in $r_n^{(\ell)}(u)$ and $\delta_n^{(\ell)}(u)$, the exponential functions and the error functions modulate the polynomial $p_n^{(\ell)}(u)$ for the sole purpose of transforming the parity to $(-1)^{n+1}$.

4. The behavior of $\overline{p}_n^{(\ell)}(u)$ near $u = 0$

In the previous section, the functions $R_n^{(\ell)}(u)$ were grouped into $p_n^{(\ell)}(u)$ and $\overline{p}_n^{(\ell)}(u)$ and their expressions in elemental functions obtained. The functions $p_n^{(\ell)}(u)$ are simple polynomials that behave like power functions u^n. The functions $\overline{p}_n^{(\ell)}(u)$ have more complex expressions. But, the functions $\overline{p}_n^{(\ell)}(u)$ approach $p_n^{(\ell)}(u)$ when u is large. In this section, we explore the behavior of $\overline{p}_n^{(\ell)}(u)$ when u is near zero.

Another way to group the functions $R_n^{(\ell)}(u)$ is based on their parity. If $R_n^{(\ell)}(u)$ is an odd function, it will be referred to here as $o_n^{(\ell)}(u)$. The corresponding even function is $\overline{o}_n^{(\ell)}(u)$. The odd and even structured power functions are defined respectively as

$$o_n^{(\ell)}(u) = \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^1 \left[e^{-(y-ux)^2} - e^{-(y+ux)^2}\right] y^n x^\ell \, dx\, dy, \tag{33}$$

$$\overline{o}_n^{(\ell)}(u) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \int_0^1 \left[e^{-(y-ux)^2} + e^{-(y+ux)^2} \right] y^n x^{\ell} \, dx \, dy. \tag{34}$$

From equations (33) and (34), the dependence of the associate index ℓ on $R_n^{(\ell)}(u)$ can be derived. It was found that $e^{-(y-ux)^2} \geqslant e^{-(y+ux)^2}$ for $u > 0$. It follows that $[e^{-(y-ux)^2} \pm e^{-(y+ux)^2}] > 0$. Since $0 < x < 1$, the larger the ℓ the smaller the integrals, that is

$$o_n^{(\ell)}(u) > o_n^{(\ell+1)}(u) > 0, \qquad \text{for} \quad u > 0, \tag{35}$$

$$\overline{o}_n^{(\ell)}(u) > \overline{o}_n^{(\ell+1)}(u) > 0, \qquad \text{for} \quad u > 0. \tag{36}$$

According to equation (33), the odd functions $o_n^{(l)}(u)$ are always 0 when $u = 0$. The value of the even functions $\overline{o}_n^{(l)}(0)$ can be obtained by integrating equation (34) after substituting $u = 0$. It was found that

$$\overline{o}_n^{(\ell)}(0) = \frac{2\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}(\ell+1)}. \tag{37}$$

The derivative of an even function is an odd function, whereas the derivative of an odd function is an even function. The derivative relations in terms of $o_n^{(l)}(u)$ and $\overline{o}_n^{(l)}(0)$ are

$$\frac{d}{du} o_n^{(l)}(u) = n\overline{o}_{n-1}^{(l+1)}(u) \tag{38}$$

and

$$\frac{d}{du} \overline{o}_n^{(l)}(u) = n o_{n-1}^{(l+1)}(u). \tag{39}$$

Using the derivative relations equations (38) and (39) and holding the value of the functions at zero equation (37), we can obtain the Taylor expansions for the functions $\overline{o}_n^{(l)}(u)$ and $o_n^{(l)}(u)$ at $u = 0$. In the case of even functions $\overline{o}_n^{(l)}(u)$, only the derivatives of even numbers have non-zero values at $u = 0$, i.e.

$$\frac{d^{2j}}{du^{2j}} \overline{o}_n^{(\ell)}(0) = \frac{n!}{(n-2j)!} \overline{o}_{n-2j}^{(\ell+2j)}(0). \tag{40}$$

The Taylor expansion at zero for $\overline{o}_n^{(l)}(u)$ is

$$\lim_{u\to 0} \overline{o}_n^{(\ell)}(u) = \sum_{j=0}^{\lfloor n/2 \rfloor} C_n^{2j} \frac{2\Gamma\left(\frac{n+1}{2} - j\right)}{\sqrt{\pi}(\ell+2j+1)} u^{2j}. \tag{41}$$

In the case of odd functions $o_n^{(\ell)}(u)$, only the derivatives of odd numbers of $2j+1$ have a non-zero value at $u = 0$, i.e.

$$\frac{d^{2j+1}}{du^{2j+1}} o_n^{(\ell)}(0) = \frac{n!}{(n-2j-1)!} \overline{o}_{n-2j-1}^{(\ell+2j+1)}(0). \tag{42}$$

The Taylor expansion at zero for $o_n^{(\ell)}(u)$ is

$$\lim_{u\to 0} o_n^{(\ell)}(u) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} C_n^{2j+1} \frac{2\Gamma\left(\frac{n}{2} - j\right)}{\sqrt{\pi}(\ell+2j+2)} u^{2j+1}. \tag{43}$$

In order to obtain the Taylor expansion for $\overline{p}_n^{(\ell)}(u)$ at $u = 0$, we constructed $\overline{p}_n^{(\ell)}(u)$ from the even and odd functions. The parities of the function $\overline{p}_n^{(\ell)}(u)$ are always opposite to their power n, namely $\overline{p}_n^{(\ell)}(-u) = (-1)^{n+1}\overline{p}_n^{(\ell)}(u)$. Thus, we obtain

$$\overline{p}_n^{(\ell)}(u) = \frac{1+(-1)^n}{2} o_n^{(\ell)}(u) + \frac{1-(-1)^n}{2} \overline{o}_n^{(\ell)}(u). \tag{44}$$

The expansion of $\overline{p}_n^{(\ell)}(u)$ at zero is obtained by substituting equations (41) and (43) for $o_n^{(\ell)}(u)$ and $\overline{o}_n^{(\ell)}(u)$, respectively, in equation (44)

$$\lim_{u\to 0}\overline{p}_n^{(\ell)}(u) = z_{n-1}^{(\ell)}(u) = \frac{2}{\sqrt{\pi}}\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} C_n^{2i+1}\, i!\frac{u^{n-1-2i}}{n+\ell-2i}. \tag{45}$$

It is clear that all the coefficients of the polynomial functions $z_{n-1}^{(\ell)}(u)$ are positive. The polynomials $z_{n-1}^{(\ell)}(u)$ also behave like the power functions u^{n-1} since they have all the properties of the polynomials $p_{n-1}^{(\ell)}(u)$. We know that the expansion of $\overline{p}_n^{(l)}(u)$ at infinity is

$$\lim_{u\to\infty}\overline{p}_n^{(\ell)}(u) = p_n^{(\ell)}(u) = \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\frac{2^{1-2i}n!}{i!(n-2i)!}\frac{u^{n-2i}}{n+l+1-2i}. \tag{46}$$

Therefore, $\overline{p}_n^{(l)}(u)$ can be regarded as a smooth connection between the two polynomials (equations (45) and (46)). In general, $\overline{p}_n^{(l)}(u)$ behave like the nth power functions with parities of $(-1)^{n-1}$.

5. Examples and some applications in physics

The general properties of the functions $R_n^{(\ell)}(u)$ were described in the previous sections. In this section, we provide examples of these functions with a view to gaining a deeper understanding of their properties. We conclude with a discussion of some of their applications in physics.

The properties of structured power functions are similar to those of regular power functions u^n. At infinity, structured power functions are/approach simple polynomials $p_n^{(\ell)}(u)$. At $u = 0$, odd structured power functions are 0 and even structured power functions are $2\Gamma\left(\frac{n+1}{2}\right)\big/(\sqrt{\pi}(\ell+1))$. Here, we consider the structured power functions $R_n^{(\ell)}(u)$ for $n = 3$ and $\ell = 0, 1, 2$, and 3. For the sake of simplicity, we present only four of the eight functions

$$\begin{aligned}
p_3^{(2)}(u) &= \frac{u^3}{3}+\frac{3u}{4},\\
\overline{p}_3^{(2)}(u) &= \left(\frac{u^3}{3}+\frac{3u}{4}+\frac{u^{-3}}{16}\right)\mathrm{Erf}(u)+\frac{\mathrm{e}^{-u^2}}{\sqrt{\pi}}\left(\frac{u^2}{3}+\frac{7}{12}-\frac{u^{-2}}{8}\right),\\
p_3^{(3)}(u) &= \frac{2u^3}{7}+\frac{3u}{5},\\
\overline{p}_3^{(3)}(u) &= \left(\frac{2u^3}{7}+\frac{3u}{5}\right)\mathrm{Erf}(u)+\frac{\mathrm{e}^{-u^2}}{\sqrt{\pi}}\left(\frac{2u^2}{7}+\frac{16}{35}-\frac{3u^{-2}}{35}+\frac{3(\mathrm{e}^{u^2}-1)}{35u^4}\right).
\end{aligned}$$

All the odd functions for $n = 3$ are polynomials $p_3^{(\ell)}(u)$ and all the even functions are non-polynomials $\overline{p}_3^{(\ell)}(u)$. For $u > 0$, $R_3^{(\ell)}(u)$ are monotonic increasing functions. As shown in figure 1, all the eight functions are similar to power functions. The odd functions $p_3^{(\ell)}(u)$ and even functions $\overline{p}_3^{(\ell)}(u)$ converge when u is large. Though there are singular terms in $\overline{p}_3^{(\ell)}(u)$, the functions $\overline{p}_3^{(\ell)}(u)$ behave well at $u = 0$. The larger the ℓ, the smaller the functions.

The structured power functions have simple properties even though their expressions appear complex. For example, the expansion of $\overline{p}_4^{(3)}(u)$ has the form

$$\overline{p}_4^{(3)}(u) = \left(\frac{u^4}{4}+u^2+\frac{3}{8}-\frac{3u^{-4}}{64}\right)\mathrm{Erf}(u)+\frac{\mathrm{e}^{-u^2}}{\sqrt{\pi}}\left(\frac{u^3}{4}+\frac{7u}{8}+\frac{u^{-1}}{16}+\frac{3u^{-3}}{32}\right), \tag{47}$$

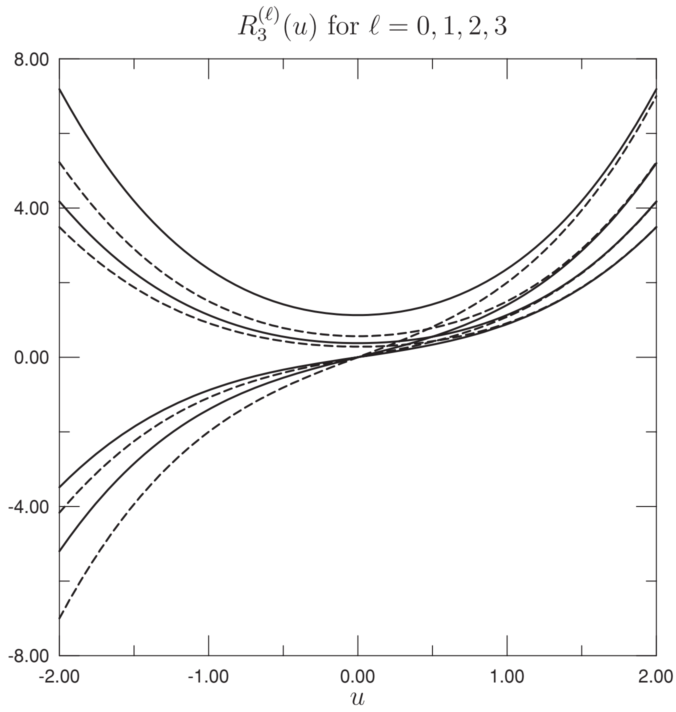

Figure 1. The curves of the functions $R_3^{(\ell)}(u)$ are like those of regular power functions. The even and the odd functions converge in pairs at positive large us. All the odd functions are the polynomials $p_3^{(\ell)}(u)$, and all the even functions are the non-polynomials $\overline{p}_3^{(\ell)}(u)$. The functions $q_3^{(\ell)}(u)$ (solid line) and $\overline{q}_3^{(\ell)}(u)$ (dashed line) converge at positive large us. In the area of convergence (the upper-right corner of the graph), ℓ equals 0, 1, 2 and 3. The larger the ℓ, the closer it is to the u-axis.

which looks complicated, but it is not. At zero point, $\overline{p}_4^{(3)}(0)$ is 0 though it appears that the function has a singularity at $u = 0$ because it includes terms like u^{-4}, u^{-3} and u^{-1}. As can be seen in figure 2, $\overline{p}_4^{(3)}(u)$ is an odd function with an even power $n = 4$. At infinity, the function $\overline{p}_4^{(3)}(u)$ approaches $p_4^{(3)}(u)$. At $u = 0$, it approaches $z_3^{(3)}(u) = \frac{8}{\sqrt{\pi}}\left(\frac{u^3}{7} + \frac{u}{5}\right)$ from equation (45). The structured power function $\overline{p}_4^{(3)}(u)$ equation (47) is merely a smooth connection between the two simple polynomials

$$p_4^{(3)}(u) = \frac{u^4}{4} + u^2 + \frac{3}{8}, \tag{48}$$

$$z_3^{(3)}(u) = \frac{8}{\sqrt{\pi}}\left(\frac{u^3}{7} + \frac{u}{5}\right). \tag{49}$$

The standard approach to calculating the derivative of $\overline{p}_4^{(3)}(u)$ from equation (47) is extremely tedious. The derivative of $\overline{p}_4^{(3)}(u)$ is simply $4\overline{p}_3^{(4)}(u)$ based on the derivative property of the structured power function. In conclusion, the properties of structured power functions are simple and behave like those of regular power functions u^n though they appear complicated when expressed in common elemental functions.

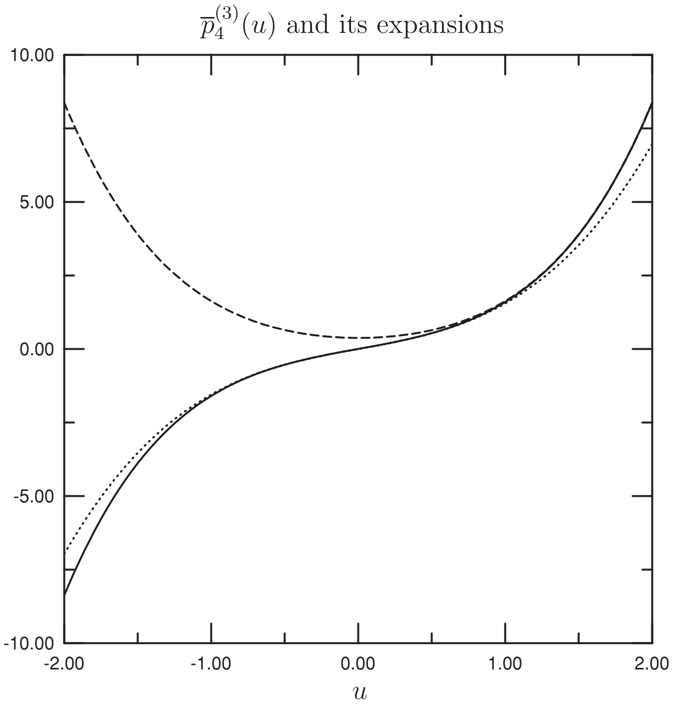

Figure 2. The function $\overline{p}_4^{(3)}(u)$ (solid line) is the smooth connection between the two polynomials $z_3^{(3)}(u)$ and $p_4^{(3)}(u)$. The function $\overline{p}_4^{(3)}(u)$ converges to $\frac{8}{\sqrt{\pi}}(\frac{u^3}{7}+\frac{u}{5})$ (dotted line) when u approaches 0 and to $\frac{u^4}{4}+u^2+\frac{3}{8}$ (dashed line) when u approaches positive infinity.

Structured power functions emerged during the calculation of arbitrary order Fokker–Planck coefficients for both Coulomb interactions and hard sphere interactions. The unified Fokker–Planck coefficients are expressed as [4]

$$\left\langle \Delta v_{\|}^{n-2(j+k)} \Delta v_{\perp 1}^{2j} \Delta v_{\perp 2}^{2k} \right\rangle = \nu_0 (-\alpha v_{\text{th}})^n Q_{k,n+\beta}^{j,(n)}(u), \tag{50}$$

where

$$Q_{k,n}^{j,(\ell)}(u) = \sum_{i=0}^{j+k} \frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{i!(j+k-i)!\pi} (-1)^{j+k-i} q_n^{(\ell-2i)}(u). \tag{51}$$

The physical parameter ν_0 represents a constant of collision frequency, α the mass coefficient of the particles and v_{th} the thermal velocity of the bath particles. The parameter β equals -3 for Coulomb interactions and 1 for hard sphere interactions. It appears only in the power index for structured power functions $q_{n+\beta}^{(\ell)}(u)$. For the same Fokker–Planck coefficients, the results for hard sphere interactions are always four orders higher than those for Coulomb interactions.

In order to calculate the speed moments or super potentials (equation (2)), we introduce a unit integral

$$\frac{n+1}{2} \int_0^{\pi} \int_0^{2\pi} \sin^{n-1}\left(\frac{\theta}{2}\right) \frac{1}{4\pi} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi = 1. \tag{52}$$

Combining the unit integral with equation (2), we can express the speed moments in a five-fold integral

$$M_n(v) = \frac{n+1}{2}\int [g\sin(\theta/2)]^{n-1} f_M(v_b)\frac{g}{4\pi}\,d\Omega\, d\mathbf{v}_b, \tag{53}$$

where $d\Omega = \sin\theta\, d\theta\, d\varphi$ is the differential of solid scattering angle and $g = |\mathbf{v} - \mathbf{v}_b|$ is the relative speed. Equation (53) can be interpreted as a collision integral of hard sphere interactions. Here θ is the scattering angle, and $\Delta v = g\sin(\theta/2)$ is the velocity change in a collision. Equation (53) can be transformed into the integral

$$M_n(v) = \frac{n+1}{2}\int (\Delta v)^{n-1} K(\mathbf{v}, \Delta\mathbf{v})\, d\Delta\mathbf{v}, \tag{54}$$

where

$$K(\mathbf{v}, \Delta\mathbf{v}) = \frac{\pi n_b v_{\rm th}}{(\sqrt{\pi} v_{\rm th})^3}\, e^{-(\Delta v + \mathbf{v}\cdot\frac{\Delta\mathbf{v}}{\Delta v})^2/v_{\rm th}^2}\frac{v_{\rm th}}{\Delta v} \tag{55}$$

is the kernel [4, 15, 16] for hard sphere interactions. Introducing the dimensionless variables $y = \Delta v/v_{\rm th}$, $u = v/v_{\rm th}$ and $x = \mathbf{v}\cdot\Delta\mathbf{v}/(v\Delta v)$, we obtain from equation (54)

$$M_n(v) = n_b v_{\rm th}^n \frac{n+1}{2}\int_0^\infty\int_{-1}^1 \frac{2}{\sqrt{\pi}} e^{-(y+ux)^2} y^n\, dx\, dy. \tag{56}$$

The speed moments or super potentials (equation (2)) are reduced to a simple expression

$$M_n(v) = n_b v_{\rm th}^n \frac{n+1}{2} q_n^{(0)}(u), \tag{57}$$

where n_b is the density of the background particles.

In terms of structured power functions, the transfer moments (equation (4)) can be expressed as

$$G_n(v) = \nu_0\left(\frac{m}{2}\alpha^2 v_{\rm th}^2\right)^n \sum_{k=0}^n C_n^k\left(-\frac{2u}{\alpha}\right)^k q_{2n-k+\beta}^{(k)}(u). \tag{58}$$

Fokker–Planck coefficients (equation (50)), super potentials (equation (57)) and transfer moments (equation (58)) are all expressed in the functions $q_n^{(\ell)}(u)$. Without structured power functions, it is impossible to express these physical results in their arbitrary orders. These results have been verified by comparing them to standard results for lower orders. The zeroth-order Fokker–Planck coefficient is the collision frequency. When $n = 0$ and $\beta = 1$, we obtain the well-known collision frequency from equation (50) for hard sphere interactions [17, 18]

$$\langle 1\rangle = \nu_0 q_1^{(0)}(u) = \nu_0\left[\left(u + \frac{1}{2u}\right)\mathrm{Erf}(u) + \frac{e^{-u^2}}{\sqrt{\pi}}\right]. \tag{59}$$

The speed moments $M_1(v)$, $M_3(v)$ and $M_5(v)$ from equation (57) were found to be in agreement with the results from Shoub [5].

The first three transfer moments, $G_1(v)$, $G_2(v)$ and $G_3(v)$, from equation (58) are in agreement with the results from Andersen and Shuler [10]. When the transfer moments are expressed in the form of regular functions, they are very complex. Mathematica code has been employed for the tedious verification of the transfer moment $G_3(v)$.

The functions $q_n^{(\ell)}(u)$ are the only ones that appear in physical applications. We have not found any physical applications for $\overline{q}_n^{(\ell)}(u)$. However, it is necessary to introduce $\overline{q}_n^{(\ell)}(u)$ for a better understanding of these kinds of functions.

The functions $q_n^{(\ell)}(u)$ in equations (50) and (58) always have $(-1)^{n+\ell} = (-1)^\beta = -1$ since β is 1 for hard sphere interactions and −3 for Coulomb interactions. We may replace $q_n^{(\ell)}(u)$

with $\overline{p}_n^{(\ell)}(u)$ in equations (50) and (58). This is the reason that Fokker–Planck coefficients and transfer moments are so complex when expressed in common elemental functions. However, the properties of $\overline{p}_n^{(\ell)}(u)$ are similar to those of simple power functions u^n except for parity. The parity for $\overline{p}_n^{(\ell)}(u)$ is $(-1)^{n+1}$ rather than $(-1)^n$.

6. Summary and discussion

The formulae for the calculation of the integrals (equation (5)) are provided in equations (15) and (16). The results of the integration $R_n^{(\ell)}(u)$ were found to have interesting properties. In general, the functions $R_n^{(\ell)}(u)$ behave like simple power functions u^n. $R_n^{(\ell)}(u)$ are called structured power functions because their curves and derivatives are similar to those of u^n. The functions $R_n^{(\ell)}(u)$ can have an odd and even parities. When the parity is $(-1)^n$, which is consistent with the power n, $R_n^{(\ell)}(u)$ are simple polynomials $p_n^{(\ell)}(u)$. Otherwise, $R_n^{(\ell)}(u)$ are $\overline{p}_n^{(\ell)}(u)$, which have very complex expressions in common elemental functions.

Structured power functions are used to present physical results of arbitrary high orders. These results include the Fokker–Planck coefficients (equation (50)), super potentials (equation (57)) and transfer moments (equation (58)). Structured power functions allow physical results to be presented in simple and meaningful expressions.

In this paper, structured power functions $R_n^{(\ell)}(u)$ have limited to non-negative powers. However, in the case of negative n, such functions have physical applications. It was found that $q_{-1}^{(0)}(u)$ and $q_{-2}^{(1)}(u)$ are related to the diffusion and friction coefficients for Coulomb interactions [4]. However, the associated integrals (equation (5)) were found to diverge at $y = 0$. Therefore, the functions $R_n^{(\ell)}(u)$ for negative n cannot be defined rigorously by equation (5). The definition of the negative power $R_n^{(\ell)}(u)$ will be considered in future work.

Acknowledgment

I would like to thank Dr Yu Zhang for useful discussions on the subject and the help in preparing the figures.

References

[1] Chapman S and Cowling T G 1970 *The Mathematical Theory of Non-Uniform Gas* (Cambridge: Cambridge University Press) chapter 5
[2] Spitzer L Jr 1962 *Physics of Fully Ionized Gases* (New York: Wiley) chapter 5
[3] Chang Y B 2003 *Phys. Plasmas* **10** 4645
[4] Chang Y B 2006 *Phys. Plasmas* **13** 012302
[5] Shoub E C 1987 *Phys. Fluids* **30** 1340
[6] Rosenbluth M N, MacDonald W and Judd D 1957 *Phys. Rev.* **107** 1
[7] Li C K and Petrasso R D 1993 *Phys. Rev. Lett.* **70** 3063
[8] Chang Y B and Li D 1996 *Phys. Rev.* E **53** 3999
[9] Hoare M R 1970 *Adv. Chem. Phys.* **20** 135
[10] Andersen K and Shuler K E 1964 *J. Chem. Phys.* **40** 633
[11] Wolfram S 1991 *Mathematica, A System for Doing Mathematics by Computer* (New York: Addison-Wesley) p 578
[12] Arfken G 1966 *Mathematical Methods for Physicists* (New York: Academic) p 502
[13] Roman S and Rota G C 1978 *Adv. Math.* **27** 95
[14] He M X and Ricci P E 2002 *J. Comput. Appl. Math.* **139** 231
[15] Berman P R, Haverkort J E and Woerdman J P 1986 *Phys. Rev.* A **34** 4647
[16] Yamaguchi T and Kimura Y 2001 *J. Chem. Phys.* **114** 3029
[17] Sone Y, Takata S and Wakabayashi M 1994 *Phys. Fluids* **6** 1914
[18] Whipple E C Jr 1972 *Phys. Fluids* **15** 988